

Find the general solution of $4x^2 y'' + 8xy' + y = \frac{16}{\sqrt{x}}$.

$$g = \frac{16x^{-\frac{1}{2}}}{4x^2} = 4x^{-\frac{5}{2}}$$

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$$4r^2 + 4r + 1 = 0$$

$$(2r+1)^2 = 0$$

$$r = -\frac{1}{2}, -\frac{1}{2}$$

NOTE: $x > 0$

$$y_1 = x^{-\frac{1}{2}}, y_2 = x^{-\frac{1}{2}} \ln x$$

$$W = \begin{vmatrix} x^{-\frac{1}{2}} & x^{-\frac{1}{2}} \ln x \\ -\frac{1}{2}x^{-\frac{3}{2}} & -\frac{1}{2}x^{-\frac{3}{2}} \ln x + x^{-\frac{3}{2}} \end{vmatrix} = x^{-2}$$

$$y_p = -x^{-\frac{1}{2}} \int \frac{4x^{-\frac{5}{2}} x^{-\frac{1}{2}} \ln x}{x^{-2}} dx + x^{-\frac{1}{2}} \ln x \int \frac{4x^{-\frac{5}{2}} x^{-\frac{1}{2}}}{x^{-2}} dx$$

$$= -x^{-\frac{1}{2}} \int 4x^{-1} \ln x dx + x^{-\frac{1}{2}} \ln x \int 4x^{-1} dx$$

$$= -x^{-\frac{1}{2}} (2(\ln x)^2) + x^{-\frac{1}{2}} \ln x (4 \ln |x|) \quad (x > 0)$$

$$= 2x^{-\frac{1}{2}} (\ln x)^2$$

$$y = 2x^{-\frac{1}{2}} (\ln x)^2 + C_1 x^{-\frac{1}{2}} + C_2 x^{-\frac{1}{2}} \ln x$$

(2) EACH

$y = x$ is a solution of

$$x^2 y'' + (2x^2 - 2x)y' + (2 - 2x)y = 0.$$

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$y = x^2$ is a particular solution of

$$x^2 y'' + (2x^2 - 2x)y' + (2 - 2x)y = 2x^3.$$

Solve the initial value problem

$$x^2 y'' + (2x^2 - 2x)y' + (2 - 2x)y = -8x^3, \quad y(1) = 2, \quad y'(1) = -14.$$

$$y_2 = vx$$

$$y_2' = v'x + v$$

$$y_2'' = v''x + 2v'$$

$$x^2(xv'' + 2v')$$

$$+ (2x^2 - 2x) \quad xv' + v$$

$$+ (2 - 2x) \quad xv$$

$$= x^3 v'' + 2x^3 v' = 0$$

$$v'' + 2v' = 0 \quad u = v'$$

$$u' + 2u = 0$$

$$\frac{1}{u} du = -2 dx$$

$$\ln|u| = -2x$$

$$u = v' = e^{-2x}$$

$$v = -\frac{1}{2}e^{-2x}$$

$$y_2 = -\frac{1}{2}xe^{-2x} \text{ or } xe^{-2x}$$

$$y = -4x^2 + Ax + Bxe^{-2x}$$

$$y' = -8x + A + Be^{-2x} - 2Bxe^{-2x}$$

$$2 = -4 + A + Be^{-2}$$

$$-14 = -8 + A + Be^{-2} - 2Be^{-2}$$

$$= -8 + A - Be^{-2}$$

$$-12 = -12 + 2A \rightarrow A = 0 \quad \textcircled{1}$$

$$2 = -4 + Be^{-2} \rightarrow B = 6e^2$$

$$y = -4x^2 + 6xe^{2-2x}$$

①

② EACH

EXCEPT AS NOTED

Find the general solution of $y''' + y'' - 2y = 2e^t - 4e^t \cos t$.

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$$r^3 + r^2 - 2 = 0$$

$$\begin{array}{r|rrrr} 1 & 1 & 1 & 0 & -2 \\ & & 1 & 2 & 2 \\ \hline & 1 & 2 & 2 & 0 \end{array}$$

$$r^2 + 2r + 2 = 0$$

$$r = -1 \pm i$$

$$y_h = c_1 e^t + c_2 e^{-t} \cos t + c_3 e^{-t} \sin t \quad (3)$$

$$y_p = A t e^t + B e^t \cos t + C e^t \sin t$$

$$y_p' = (A t + A) e^t + (B + C) e^t \cos t + (C - B) e^t \sin t \quad (3)$$

$$y_p'' = (A t + 2A) e^t + 2C e^t \cos t - 2B e^t \sin t \quad (3)$$

$$y_p''' = (A t + 3A) e^t + (-2B + 2C) e^t \cos t + (-2C - 2B) e^t \sin t \quad (3)$$

$$y_p''' + y_p'' - 2y_p = 5A e^t + (-4B + 4C) e^t \cos t + (-4C - 4B) e^t \sin t$$

$$5A = 2$$

$$A = \frac{2}{5}$$

$$-4B + 4C = -4$$

$$-4C - 4B = 0$$

$$-B + C = -1$$

$$B + C = 0$$

$$2C = -1$$

$$C = -\frac{1}{2}$$

$$B = \frac{1}{2}$$

$$y = \frac{2}{5} t e^t + \frac{1}{2} e^t \cos t - \frac{1}{2} e^t \sin t + c_1 e^t + c_2 e^t \cos t + c_3 e^t \sin t$$

(2) EACH
EXCEPT AS NOTED

Use elimination (as shown in lecture) to solve the system

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$$(2D+1)[x] - (3D+4)[y] = 4t$$

$$(3D+2)[x] - (5D+8)[y] = 2t$$

$$x = \frac{4t^2 + 2t}{t^2 + 2t} \left[\frac{2 + 4c_1 + c_2 e^{-3t}}{+ c_1 + c_2 e^{-3t}} \right]$$

$$\begin{aligned} & \left[(3D+2)(3D+4) + (2D+1)(5D+8) \right] [y] = (3D+2)[4t] - (2D+1)[2t] \\ & (D^2+3D)[y] = 12+8t - (4+2t) = 8+6t \\ & r^2+3r=0 \rightarrow r=0, -3 \end{aligned}$$

$$y_h = c_1 + c_2 e^{-3t}$$

$$y_p = (At+B)t = At^2 + Bt$$

$$y_p' = 2At + B$$

$$y_p'' = 2A$$

$$y_p'' + 3y_p' = 6At + (2A+3B) = 8+6t$$

$$6A=6$$

$$2A+3B=8$$

$$A=1$$

$$2+3B=8$$

$$B=2$$

$$y = t^2 + 2t + c_1 + c_2 e^{-3t}$$

$$[(5D+8)(2D+1) - (3D+4)(3D+2)][x] = (5D+8)[4t] - (3D+4)[2t]$$

$$(D^2+3D)[x] = 20+32t - (6+8t) = 14+24t$$

$$x_h = k_1 + k_2 e^{-3t}$$

$$x_p = Ct^2 + Dt$$

$$x_p'' + 3x_p' = 6Ct + (2C+3D) = 14+24t$$

$$6C=24 \quad 2C+3D=14$$

$$C=4$$

$$8+3D=14$$

$$D=2$$

$$x = 4t^2 + 2t + k_1 + k_2 e^{-3t}$$

$$\begin{aligned} & 2(8t + 2 - 3k_2 e^{-3t}) \\ & + (4t^2 + 2t + k_1 + k_2 e^{-3t}) \\ & - 3(2t + 2 - 3c_2 e^{-3t}) \\ & - 4(t^2 + 2t + c_1 + c_2 e^{-3t}) \end{aligned}$$

$$= 4t + (k_1 - 4c_1 - 2) + (-5k_2 + 5c_2) e^{-3t}$$

$$k_1 - 4c_1 - 2 = 0$$

$$-5k_2 + 5c_2 = 0$$

$$k_1 = 4c_1 + 2$$

$$k_2 = c_2$$